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2008 J. Phys. A: Math. Theor. 41 304001

(http://iopscience.iop.org/1751-8121/41/30/304001)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 304001 (12pp) doi:10.1088/1751-8113/41/30/304001

Fock representation of the renormalized higher powers of White noise and the centreless Virasoro (or Witt)–Zamolodchikov–w_∞*-Lie algebra

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Received 30 November 2007, in final form 20 January 2008
Published 15 July 2008
Online at stacks.iop.org/JPhysA/41/304001

Abstract
The identification of the *-Lie algebra of the renormalized higher powers of White noise (RHPWN) and the analytic continuation of the second quantized centreless Virasoro (or Witt)–Zamolodchikov–$w_\infty *$-Lie algebra of conformal field theory and high-energy physics, was recently established in [5] based on results obtained in [3] and [4]. In the present paper, we show how the RHPWN Fock kernels must be truncated in order to be positive semi-definite and we obtain a Fock representation of the two algebras. We show that the truncated renormalized higher powers of White noise (TRHPWN) Fock spaces of order $\geqslant 2$ host the continuous binomial and beta processes.

PACS numbers: 02.20.Sv, 02.50.Ey, 11.10.Nx, 11.25.Hf
Mathematics Subject Classification: 60H40, 81S05, 81T30, 81T40

1. Content outline

The material presented in this paper is organized as follows:

In section 2, we present the basic theory of the *-Lie algebra of the renormalized higher powers of White noise (RHPWN) whose commutation relations are obtained through a new renormalization of the powers of the Dirac delta function introduced in [3] and [4]. We also describe the recently discovered connection between the RHPWN *-Lie algebra and the Virasoro–Zamolodchikov–$w_\infty *$-Lie algebra of conformal field theory and high-energy physics (cf [5]).

In section 3, we give a definition of the action of the RHPWN *-Lie algebra generators B_k^n where $n, k \geqslant 0$ on the Fock vacuum vector Φ that is based on White noise and norm compatibility arguments.

In section 4, we define the nth order *-Lie subalgebra $\mathcal{L}_n$ of the RHPWN *-Lie algebra, generated by B_0^n and B_n^0 where $n \geqslant 1$. The no-go theorems for the Fock representation of the RHPWN *-Lie algebra proved in [2, 4, 8] are shown to extend to $\mathcal{L}_n$ even with the new renormalization of the Dirac delta function described in section 2.

In section 5, for each $n \geqslant 1$ we describe the singular terms appearing in the nth order Fock kernel $\langle (B_0^n)^k \Phi, (B_0^n)^k \Phi \rangle$ where $k \geqslant 0$. These terms prevent the kernel from being positive semi-definite. To eliminate the singular terms a truncation scheme is introduced.

In section 6, we describe the general Fock space construction method of [16].

In section 7, using the truncation scheme introduced in section 5 and the Fock space construction method described in section 6 we explicitly compute the inner product associated with the nth order Fock space $\mathcal{F}_n$ corresponding to the nth order *-Lie subalgebra $\mathcal{L}_n$ of the RHPWN *-Lie algebra.

In section 8, for each $n \geqslant 1$ we give the exact representation of the RHPWN generators B_0^n, B_n^0 and B_{n-1}^{n-1} as operators acting on the nth order Fock space $\mathcal{F}_n$. The way to obtain a Fock representation of an arbitrary RHPWN generator B_k^n is also described.

In sections 9 and 10, in the spirit of quantum probability we show that the self-adjoint operator process corresponding to $B_0^n + B_n^0$ can be identified with classical Brownian motion if $n = 1$, and the continuous binomial or beta processes if $n \geqslant 2$.

2. The theory of the renormalized higher powers of White noise

Giving meaning to the powers of the creation and annihilation densities (quantum White noise) is an old and important problem in quantum field theory. For a general description of the problem and its connections with classical probability we refer to [1]. The developments up to this point, can be described as follows. Let a_t and $a_s^\dagger$ be the standard boson White noise functionals with commutator

$$\left[a_t, a_s^\dagger\right] = \delta(t-s)\cdot 1,$$

where δ is the Dirac delta function. As shown in [3, 4], choosing as test function space the space of piecewise continuous compactly supported functions $f : \mathbb{R} \to \mathbb{C}$ that vanish at zero, on the vector space generated by the symbols

$$B_k^n(f) = \int_{\mathbb{R}} f(s) a_s^{\dagger^n} a_s^k \,\mathrm{d}s;\; n, k \in \{0, 1, 2, \ldots\} \tag{2.1}$$

with $B_0^0(f) = \int_{\mathbb{R}} f(s)\,\mathrm{d}s$, one can define a *-Lie algebra structure with involution

$$\left(B_k^n(f)\right)^* = B_n^k(\bar{f}) \tag{2.2}$$

and with Lie brackets uniquely defined by the prescriptions

$$\left[B_k^n(f), B_K^N(g)\right] = \int_{\mathbb{R}}\int_{\mathbb{R}} f(s) g(t) \left[a_s^{\dagger^n} a_s^k, a_t^{\dagger^N} a_t^K\right] \mathrm{d}s\,\mathrm{d}t,$$

where $n, k \in \{0, 1, 2, \ldots\}$, together with the renormalization rule for the higher powers of the Dirac delta function

$$\delta^l(t-s) = \delta(s)\delta(t-s), \qquad l = 2, 3, \ldots. \tag{2.3}$$

Furthermore one can prove that the above prescriptions lead to the following Lie brackets:

$$\left[B_k^n(g), B_K^N(f)\right]_{\mathrm{RHPWN}} = (kN - Kn)\, B_{k+K-1}^{n+N-1}(gf), \tag{2.4}$$

where for $n < 0$ and/or $k < 0$ we define $B_k^n(f) = 0$. The relations (2.4) will be called the RHPWN commutation relations. It was also proved in [3] and [4] that, for $n, N \geqslant 2$ and

$k, K \in \mathbb{Z}$ the White noise operators (see [3] and [4] for a precise definition of the integral and the integrand below)

$$\hat{B}_k^n(f) = \int_{\mathbb{R}} f(t)\, \mathrm{e}^{\frac{k}{2}(a_t - a_t^\dagger)} \left(\frac{a_t + a_t^\dagger}{2}\right)^{n-1} \mathrm{e}^{\frac{k}{2}(a_t - a_t^\dagger)}\, \mathrm{d}t$$

satisfy the commutation relations

$$\left[\hat{B}_k^n(g), \hat{B}_K^N(f)\right]_{w_\infty} = ((N-1)k - (n-1)K)\hat{B}_{k+K}^{n+N-2}(gf) \tag{2.5}$$

of the centreless Virasoro (or Witt)–Zamolodchikov–w_∞ Lie algebra of conformal field theory which becomes a *-Lie algebra with involution $\left(\hat{B}_k^n(f)\right)^* = \hat{B}_{-k}^n(\bar{f})$. In particular, for $n = N = 2$ we obtain

$$\left[\hat{B}_k^2(g), \hat{B}_K^2(f)\right]_{w_\infty} = (k - K)\, \hat{B}_{k+K}^2(gf)$$

which are the commutation relations of the Virasoro algebra. The analytic continuation $\left\{\hat{B}_z^n(f); n \geqslant 1, z \in \mathbb{C}\right\}$ of the centreless Virasoro (or Witt)–Zamolodchikov–w_∞ Lie algebra, and the RHPWN Lie algebra with commutator $[\cdot,\cdot]_{\mathrm{RHPWN}}$ have recently been identified (cf [5]) thus providing a connection between quantum probability, conformal field theory and high-energy physics. This connection must be further explored.

Notation 1. *In what follows, for all integers n, k we will use the notation $B_k^n = B_k^n(\chi_I)$ where I is some fixed subset of $\mathbb{R}$ of finite measure $\mu = \mu(I) > 0$.*

3. The action of the RHPWN operators on the Fock vacuum

To formulate a reasonable definition of the action of the RHPWN operators on the Fock vacuum vector Φ, we go to the level of White noise. The proof of the following lemma can be found in [6].

Lemma 1. *For all $t \geqslant s \geqslant 0$ and $n \in \{0, 1, 2, \ldots\}$*

$$\left(a_t^\dagger\right)^n (a_s)^n = \sum_{k=0}^{n} s_{n,k} \left(a_t^\dagger a_s\right)^k \delta^{n-k}(t-s),$$

where $s_{n,k}$ are the Stirling numbers of the first kind with $s_{0,0} = 1$ and $s_{0,k} = s_{n,0} = 0$ for all $n, k \geqslant 1$.

Proposition 1. *For all integers $n \geqslant k \geqslant 0$ and for all test functions f in the test function space defined before (2.1), one has*

$$B_k^n(f) = \int_{\mathbb{R}} f(t) \left(a_t^\dagger\right)^{n-k} \left(a_t^\dagger a_t\right)^k \mathrm{d}t.$$

Proof. For $n \geqslant k$ we can write $\left(a_t^\dagger\right)^n (a_s)^k = \left(a_t^\dagger\right)^{n-k} \left(a_t^\dagger\right)^k (a_s)^k$. Multiplying both sides by $f(t)\delta(t-s)$ and then taking $\int_{\mathbb{R}}\int_{\mathbb{R}} \ldots \mathrm{d}s\, \mathrm{d}t$ of both sides of the resulting equation we obtain

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f(t)\left(a_t^\dagger\right)^n (a_s)^k \delta(t-s)\, \mathrm{d}s\, \mathrm{d}t = \int_{\mathbb{R}}\int_{\mathbb{R}} f(t)\left(a_t^\dagger\right)^{n-k}\left(a_t^\dagger\right)^k (a_s)^k \delta(t-s)\, \mathrm{d}s\, \mathrm{d}t$$

which, after applying (2.1) to its left and lemma 1 to its right-hand side, yields

$$\begin{aligned} B_k^n(f) &= \sum_{m=0}^{k} s_{k,m} \int_{\mathbb{R}} \int_{\mathbb{R}} f(t) \left(a_t^\dagger\right)^{n-k} \left(a_t^\dagger a_s\right)^m \delta^{k-m+1}(t-s)\, \mathrm{d}s\, \mathrm{d}t \\ &= s_{k,k} \int_{\mathbb{R}} \int_{\mathbb{R}} f(t) \left(a_t^\dagger\right)^{n-k} \left(a_t^\dagger a_s\right)^k \delta(t-s)\, \mathrm{d}s\, \mathrm{d}t \\ &\quad + \sum_{m=0}^{k-1} s_{k,m} \int_{\mathbb{R}} \int_{\mathbb{R}} f(t) \left(a_t^\dagger\right)^{n-k} \left(a_t^\dagger a_s\right)^m \delta(s)\delta(t-s)\, \mathrm{d}s\, \mathrm{d}t \\ &= s_{k,k} \int_{\mathbb{R}} f(t) \left(a_t^\dagger\right)^{n-k} \left(a_t^\dagger a_t\right)^k \mathrm{d}t + 0 \\ &= \int_{\mathbb{R}} f(t) \left(a_t^\dagger\right)^{n-k} \left(a_t^\dagger a_t\right)^k \mathrm{d}t, \end{aligned}$$

where we have repeatedly used the renormalization rule (2.3), the condition $f(0) = 0$, and $s_{k,k} = 1$. □

Proposition 1 suggests that for all $n, k \in \{0, 1, 2, \ldots\}$ and test functions f, we define

$$B_k^n(f)\Phi = \begin{cases} 0 & \text{if} \quad n < k \text{ or } n \cdot k < 0 \\ B_0^{n-k}(f\sigma_k)\Phi & \text{if} \quad n > k \geqslant 0 \\ \displaystyle\int_{\mathbb{R}} f(t)\rho_k(t)\, \mathrm{d}t\, \Phi & \text{if} \quad n = k, \end{cases} \tag{3.1}$$

where σ_k and ρ_k are complex valued functions. Through norm compatibility arguments (cf [7] for details) we can show that for all $n \in \{0, 1, 2, \ldots\}$, $\sigma_n = \sigma_1^n$ and $\rho_n = \frac{\sigma_1^n}{n+1}$. In view of the interpretation of $a_t^\dagger$ and a_t as creation and annihilation densities respectively, it makes sense to assume that in the definition of the action of B_k^n on Φ it is only the difference $n - k$ that matters. Therefore, we take the function σ_1 appearing in (3.1) to be identically equal to 1 and we arrive to the following definition of the action of the RHPWN operators on the Fock vacuum vector Φ.

Definition 1. *For all test functions f we define*

$$B_k^n(f)\Phi = \begin{cases} 0 & \text{if} \quad n < k \text{ or } n \cdot k < 0 \\ B_0^{n-k}(f)\Phi & \text{if} \quad n > k \geqslant 0 \\ \displaystyle\frac{1}{n+1}\int_{\mathbb{R}} f(t)\, \mathrm{d}t\, \Phi & \text{if} \quad n = k. \end{cases}$$

4. The nth order *-Lie subalgebra $\mathcal{L}_n$ of RHPWN and the Fock representation no-go theorem

Definition 2.

(i) *$\mathcal{L}_1$ is the *-Lie algebra generated by B_0^1 and B_1^0, i.e. $\mathcal{L}_1$ is the linear span of $\{B_0^1, B_1^0, B_0^0\}$.*
(ii) *$\mathcal{L}_2$ is the *-Lie algebra generated by B_0^2 and B_2^0, i.e. $\mathcal{L}_2$ is the linear span of $\{B_0^2, B_2^0, B_1^1\}$.*
(iii) *For $n \in \{3, 4, \ldots\}$, $\mathcal{L}_n$ is the *-Lie algebra generated by B_0^n and B_n^0 through repeated commutations and linear combinations. It is the linear span of the operators of the form B_y^x where $x - y = kn$, $k \in \mathbb{Z} - \{0\}$, and of the number operators B_x^x with $x \geqslant n - 1$.*

Definition 3. *A Fock representation of the nth order *-Lie subalgebra $\mathcal{L}_n$ of RHPWN is a pair $\{\mathcal{F}_n, \Phi\}$ consisting of a Hilbert (Fock) space $\mathcal{F}_n$ and a cyclic (vacuum) vector $\Phi \in \mathcal{F}_n$ on which operator analogues of the RHPWN generators can be defined so that their action on Φ is that of definition 1 and they satisfy the RHPWN commutation relations (2.4).*

We will show that if the RHPWN action on Φ is that of definition 1 then the Fock representation no-go theorems of [8] and [4] can be extended to the RHPWN *-Lie subalgebras $\mathcal{L}_n$ where $n \geqslant 3$. In the following, we will use the notation $B_k^n = B_k^n(\chi_I)$ where $I \subseteq \mathbb{R}$ is an interval and $\chi_I(x) = 1$ if $x \in I$, $\chi_I(x) = 0$ if $x \notin I$. Full details of the proofs contained in this section can be found in [7]. The crucial ingredient of the no-go theorems is the following lemma.

Lemma 2. *Let $n \geqslant 3$ and suppose that a Fock space representation $\{\mathcal{F}_n, \Phi\}$ of $\mathcal{L}_n$ exists. Then it contains both $B_0^n\Phi$ and $B_0^{2n}\Phi$.*

Proof. For simplicity we restrict to a single interval I of positive measure $\mu = \mu(I)$. We have $B_n^0 B_0^n \Phi = n\mu\Phi$, $B_n^0\left(B_0^n\right)^2\Phi = (2n\mu + n^3(n-1))B_0^n\Phi$ and $B_n^0\left(B_0^n\right)^3\Phi = 3n(\mu + n^2(n-1))\left(B_0^n\right)^2\Phi + n^4(n-1)(n-2)B_0^{2n}\Phi$. Since $B_n^0\left(B_0^n\right)^3\Phi \in \mathcal{F}_n$ and $\left(B_0^n\right)^2\Phi \in \mathcal{F}_n$ it follows that $B_0^{2n}\Phi \in \mathcal{F}_n$. □

Theorem 1. *Let $n \geqslant 3$. If the test function space includes functions whose support has arbitrarily small Lebesgue measure, then $\mathcal{L}_n$ does not admit a Fock representation in the sense of definition 3.*

Proof. If a Fock representation of $\mathcal{L}_n$ existed then we should be able to define inner products of the form $\left\langle \left(aB_0^{2n} + b\left(B_0^n\right)^2\right)\Phi, \left(aB_0^{2n} + b\left(B_0^n\right)^2\right)\Phi \right\rangle$ where $a, b \in \mathbb{R}$ and the RHPWN operators are defined on the same interval I of arbitrarily small positive measure $\mu(I)$. Using the notation $\langle x \rangle = \langle \Phi, x\Phi \rangle$ this amounts to the positive semi-definiteness of the matrix

$$A = \begin{bmatrix} \left\langle B_{2n}^0 B_0^{2n} \right\rangle & \left\langle B_{2n}^0 \left(B_0^n\right)^2 \right\rangle \\ \left\langle B_{2n}^0 \left(B_0^n\right)^2 \right\rangle & \left\langle \left(B_n^0\right)^2 \left(B_0^n\right)^2 \right\rangle \end{bmatrix}.$$

Using (2.4) and definition 1 we find that $\left\langle B_{2n}^0 B_0^{2n} \right\rangle = 2n\mu(I)$, $\left\langle B_{2n}^0 \left(B_0^n\right)^2 \right\rangle = 2n^3\mu(I)$ and $\left\langle \left(B_n^0\right)^2 \left(B_0^n\right)^2 \right\rangle = 2n^2\mu(I)^2 + n^4(n-1)\mu(I)$. Thus

$$A = \begin{bmatrix} 2n\mu(I) & 2n^3\mu(I) \\ 2n^3\mu(I) & 2n^2\mu(I)^2 + n^4(n-1)\mu(I) \end{bmatrix}.$$

A is a symmetric matrix, so it is positive semi-definite if and only if its minors are nonnegative. The minor determinants of A are $d_1 = 2n\mu(I)$ which is always nonnegative, and $d_2 = 2n^3\mu(I)^2(2\mu(I) - n^2 - n^3)$ which is nonnegative if and only if $\mu(I) \geqslant \frac{n^2(n+1)}{2}$. Thus the interval I cannot be arbitrarily small. A simple approximation argument allows us to extend the conclusion from characteristic functions of intervals to arbitrary piecewise continuous functions. □

5. The *n*th order truncated RHPWN (or TRHPWN) Fock space $\mathcal{F}_n$

For each $n \geqslant 1$ the generic element of the *-Lie subalgebra $\mathcal{L}_n$ of definition 2 is B_0^n. All other elements of $\mathcal{L}_n$ are obtained by taking adjoints, commutators and linear combinations. It thus makes sense to consider $\left(B_0^n(f)\right)^k\Phi$ as basis vectors for the kth particle space of the Fock space $\mathcal{F}_n$ associated with $\mathcal{L}_n$. A calculation of the 'Fock kernel' $\langle (B_0^n)^k\Phi, (B_0^n)^k\Phi\rangle$ reveals that it is the terms containing $B_0^{2n}\Phi$ that prevent the kernel from being positive semi-definite. The $B_0^{2n}\Phi$ terms appear either directly or by applying definition 1 to terms of the form $B_y^x\Phi$ where $x-y=2n$. Since $\mathcal{L}_1$ and $\mathcal{L}_2$ do not contain B_0^2 and B_0^4 respectively, the problem exists for $n\geqslant 3$ only and the Fock spaces $\mathcal{F}_1$ and $\mathcal{F}_2$ are actually not truncated. In what follows we will compute the Fock kernels by applying definition 1 and by truncating 'singular' terms of the form $\langle (B_0^n)^k\Phi, (B_0^n)^m B_y^x\Phi\rangle$ where $nk=nm+x-y$ and $x-y=2n$, i.e. $k-m=2$. This amounts to truncating the action of the principal $\mathcal{L}_n$ number operator B_{n-1}^{n-1} on the 'number vectors' $(B_0^n)^k\Phi$, which by commutation relations (2.4) and definition 1 is of the form

$$B_{n-1}^{n-1}(B_0^n)^k\Phi=\left(\frac{\mu}{n}+kn(n-1)\right)(B_0^n)^k\Phi+\sum_{i\geqslant 1}\prod_{j\geqslant 1}c_{i,j}B_0^{\lambda_{i,j}n}\Phi$$

(where for each i not all positive integers $\lambda_{i,j}$ are equal to 1), by omitting the $\sum_{i\geqslant 1}\prod_{j\geqslant 1}c_{i,j}B_0^{\lambda_{i,j}n}\Phi$ part. We thus arrive at the following:

Definition 4. *A truncated Fock representation of the RHPWN is a Fock space representation of the RHPWN *-Lie algebra such that, for any integers $n\geqslant 1$ and $k\geqslant 0$,*

$$B_{n-1}^{n-1}(B_0^n)^k\Phi=\left(\frac{\mu}{n}+kn(n-1)\right)(B_0^n)^k\Phi,$$

i.e. the number vectors $(B_0^n)^k\Phi$ are eigenvectors of the principal $\mathcal{L}_n$ number operator B_{n-1}^{n-1} with eigenvalues $\left(\frac{\mu}{n}+kn(n-1)\right)$.

In agreement with definition 1, for $k=0$ definition 4 yields $B_{n-1}^{n-1}\Phi=\frac{\mu}{n}\Phi$.

6. Outline of the Fock space construction method

We will construct the TRHPWN Fock spaces by using the following method (cf chapter 3 of [16]):

(i) Compute

$$\|(B_0^n)^k\Phi\|^2=\langle (B_0^n)^k\Phi, (B_0^n)^k\Phi\rangle=\pi_{n,k}(\mu),$$

where for each $k=0,1,2,\ldots,\pi_{n,k}(\mu)$ is a polynomial in μ of degree k.

(ii) Using the fact that if $k\neq m$ then $\langle (B_0^n)^k\Phi, (B_0^n)^m\Phi\rangle=0$, for $a,b\in\mathbb{C}$ compute

$$\begin{aligned}\langle \mathrm{e}^{aB_0^n}\Phi, \mathrm{e}^{bB_0^n}\Phi\rangle&=\sum_{k=0}^{\infty}\frac{(\bar{a}b)^k}{(k!)^2}\langle (B_0^n)^k\Phi, (B_0^n)^k\Phi\rangle\\&=\sum_{k=0}^{\infty}\frac{(\bar{a}b)^k}{k!}\frac{\pi_{n,k}(\mu)}{k!}=\sum_{k=0}^{\infty}\frac{(\bar{a}b)^k}{k!}h_{n,k}(\mu),\end{aligned}$$

where

$$h_{n,k}(\mu)=\frac{\pi_{n,k}(\mu)}{k!}. \tag{6.1}$$

(iii) Look for a function $G_n(u,\mu)$ such that

$$G_n(u,\mu)=\sum_{k=0}^{\infty}\frac{u^k}{k!}h_{n,k}(\mu). \tag{6.2}$$

Using the Taylor expansion of $G_n(u,\mu)$ in powers of u

$$G_n(u,\mu) = \sum_{k=0}^{\infty} \frac{u^k}{k!} \frac{\partial^k}{\partial u^k} G_n(u,\mu)|_{u=0} \tag{6.3}$$

by comparing (6.3) and (6.2) we see that

$$\frac{\partial^k}{\partial u^k} G_n(u,\mu)|_{u=0} = h_{n,k}(\mu). \tag{6.4}$$

Equation (6.4) will play a fundamental role in the search for G_n in the following section.

(iv) Reduce to single intervals and extend to step functions. For $u = \bar{a}b$, assuming that

$$G_n(u,\mu) = \mathrm{e}^{\mu \hat{G}_n(u)}. \tag{6.5}$$

which is typical for 'Bernoulli moment systems' (cf chapter 5 of [16]), equation (6.2) becomes

$$\mathrm{e}^{\mu \hat{G}_n(\bar{a}b)} = \sum_{k=0}^{\infty} \frac{(\bar{a}b)^k}{k!} h_{n,k}(\mu). \tag{6.6}$$

Take the product of (6.6) over all sets I, for test functions $f = \sum_i a_i \chi_{I_i}$ and $g = \sum_i b_i \chi_{I_i}$ with $I_i \cap I_j = \oslash$ for $i \neq j$, and end up with an expression like

$$\mathrm{e}^{\int_{\mathbb{R}} \hat{G}_n(f(t)g(t))\,\mathrm{d}t} = \prod_{i,j} \left\langle \mathrm{e}^{a_i B_0^n(\chi_{I_i})}\Phi, \mathrm{e}^{b_j B_0^n(\chi_{I_j})}\Phi \right\rangle \tag{6.7}$$

which we take as the definition of the inner product $\langle \psi_n(f), \psi_n(g)\rangle_n$ of the 'exponential vectors'

$$\psi_n(f) = \prod_i \mathrm{e}^{a_i B_0^n(\chi_{I_i})}\Phi \tag{6.8}$$

of the TRHPWN Fock space $\mathcal{F}_n$. Note that $\Phi = \psi_n(0)$.

7. Construction of the TRHPWN Fock spaces $\mathcal{F}_n$

Lemma 3. *Let $n \geqslant 1$ be fixed. Then for all integers $k \geqslant 0$*

$$B_n^0 \left(B_0^n\right)^{k+1} \Phi = n(k+1) \left(\mu + k \frac{n^2(n-1)}{2}\right) \left(B_0^n\right)^k \Phi. \tag{7.1}$$

Proof. (7.1) is true for $k = 0$ since

$$B_n^0 B_0^n \Phi = \left(B_0^n B_n^0 + \left[B_n^0, B_0^n\right]\right)\Phi = n^2 \frac{\mu}{n} \Phi = n\mu\Phi.$$

Assuming (7.1) to be true for k we have

$$\begin{aligned}
B_n^0 \left(B_0^n\right)^{k+2} \Phi &= \left(B_n^0 B_0^n\right)\left(B_0^n\right)^{k+1} \Phi = \left(B_0^n B_n^0 + n^2 B_{n-1}^{n-1}\right)\left(B_0^n\right)^{k+1} \Phi \\
&= B_0^n n(k+1) \left(\mu + k \frac{n^2(n-1)}{2}\right) \left(B_0^n\right)^k \Phi + n^2 B_{n-1}^{n-1} \left(B_0^n\right)^{k+1} \Phi \\
&= \left(n(k+1) \left(\mu + k \frac{n^2(n-1)}{2}\right) + n^2 \left(\frac{\mu}{n} + (k+1)n(n-1)\right)\right) \left(B_0^n\right)^{k+1} \Phi \\
&= n(k+2) \left(\mu + (k+1) \frac{n^2(n-1)}{2}\right) \left(B_0^n\right)^{k+1} \Phi
\end{aligned}$$

which proves (7.1) to be true for $k+1$ also, thus completing the induction. □

Proposition 2. *Let $k \geqslant 1$. Then for all $n \geqslant 1$*

$$\pi_{n,k}(\mu) = \langle (B_0^n)^k \Phi, (B_0^n)^k \Phi \rangle = k! n^k \prod_{i=0}^{k-1} \left(\mu + \frac{n^2(n-1)}{2} i \right). \tag{7.2}$$

Proof. Let $n \geqslant 1$ be fixed. Let $a_k = k! n^k \prod_{i=0}^{k-1} \left(\mu + \frac{n^2(n-1)}{2} i \right)$. Then $a_1 = n\mu$ and for $k \geqslant 1, a_{k+1} = n(k+1)\left(\mu + k\frac{n^2(n-1)}{2}\right) a_k$. Similarly, let $b_k = \langle (B_0^n)^k \Phi, (B_0^n)^k \Phi \rangle$. Then $b_1 = \langle B_0^n \Phi, B_0^n \Phi \rangle = \langle \Phi, B_n^0 B_0^n \Phi \rangle = n^2 \langle \Phi, B_{n-1}^{n-1} \Phi \rangle = n^2 \frac{\mu}{n} = n\mu$ and, by lemma 3, for $k \geqslant 1$ we have that $b_{k+1} = \langle (B_0^n)^k \Phi, B_n^0 (B_0^n)^{k+1} \Phi \rangle = n(k+1)\left(\mu + k\frac{n^2(n-1)}{2}\right) b_k$. Thus $a_k = b_k$ for all $k \geqslant 1$. □

Corollary 1. *The functions $h_{n,k}$ appearing in (6.1) are given by $h_{1,k} = \mu^k$ and for $n \geqslant 2$*

$$h_{n,k} = n^k \prod_{i=0}^{k-1} \left(\mu + \frac{n^2(n-1)}{2} i \right)$$

Proof. The proof follows from proposition 2 and (6.1). □

Corollary 2. *The functions G_n appearing in (6.2) are given by $G_1(u, \mu) = \mathrm{e}^{u\mu}$ and for $n \geqslant 2$*

$$G_n(u, \mu) = \left(1 - \frac{n^3(n-1)}{2} u \right)^{-\frac{2}{n^2(n-1)}\mu} = \mathrm{e}^{-\frac{2}{n^2(n-1)}\mu \ln(1 - \frac{n^3(n-1)}{2} u)}. \tag{7.3}$$

Proof. For G_n as in the statement of this corollary, in accordance with (6.4) we have

$$\frac{\partial^k}{\partial u^k} G_n(u, \mu)|_{u=0} = n^k \prod_{i=0}^{k-1} \left(\mu + \frac{n^2(n-1)}{2} i \right).$$

□

Corollary 3. *The functions $\hat{G}_n$ appearing in (6.5) are given by $\hat{G}_1(u) = u$ and for $n \geqslant 2$*

$$\hat{G}_n(u) = -\frac{2}{n^2(n-1)} \ln \left(1 - \frac{n^3(n-1)}{2} u \right).$$

Proof. The proof follows directly from corollary 2. □

Corollary 4. *The $\mathcal{F}_n$ inner products are given by*

$$\langle \psi_1(f), \psi_1(g) \rangle_1 = \mathrm{e}^{\int_{\mathbb{R}} \bar{f}(t) g(t) \, \mathrm{d}t} \tag{7.4}$$

and for $n \geqslant 2$

$$\langle \psi_n(f), \psi_n(g) \rangle_n = \mathrm{e}^{-\frac{2}{n^2(n-1)} \int_{\mathbb{R}} \ln\left(1 - \frac{n^3(n-1)}{2} \bar{f}(t) g(t)\right) \mathrm{d}t}, \tag{7.5}$$

where $|f(t)| < \frac{1}{n}\sqrt{\frac{2}{n(n-1)}}$ and $|g(t)| < \frac{1}{n}\sqrt{\frac{2}{n(n-1)}}$.

Proof. The proof follows from (6.7) and corollary 2. □

The function G_1 and the Fock space inner product (7.4) are associated with the Heisenberg–Weyl algebra and the quantum stochastic calculus of [18]. For $n = 2$ the function G_2 and the associated Fock space inner product (7.5) have appeared in the study of the finite-difference algebra and the square of White noise algebra in [11, 12, 14, 15]. The functions G_n of (7.3) can also be found in proposition 5.4.2 of chapter 5 of [16].

Definition 5. *The nth order TRHPWN Fock space $\mathcal{F}_n$ is the Hilbert space completion of the linear span of the exponential vectors $\psi_n(f)$ of (6.8) under the inner product $\langle \cdot, \cdot \rangle_n$ of corollary 4. The full TRHPWN Fock space $\mathcal{F}$ is the direct sum of the $\mathcal{F}_n$'s.*

8. Fock representation of the TRHPWN operators

Using (6.8) and lemma 3 we have (for a detailed proof see [7]) that for all test functions f, g, h and for all $n \geqslant 1$

$$B_n^0(f)\psi_n(g) = n\int_{\mathbb{R}} f(t)g(t)\,\mathrm{d}t\,\psi_n(g) + \frac{n^3(n-1)}{2}\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0}\psi_n(g+\epsilon f g^2)$$

$$B_0^n(f)\psi_n(g) = \frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0}\psi_n(g+\epsilon f)$$

and

$$B_{n-1}^{n-1}(fg)\psi_n(h) = \frac{1}{n}\int_{\mathbb{R}} f(t)g(t)\psi_n(h) + \frac{n(n-1)}{2}\frac{\partial^2}{\partial\epsilon\partial\rho}\Big|_{\epsilon=\rho=0}(\psi_n(h+\epsilon g+\rho f(h+\epsilon g)^2) - \psi_n(h+\epsilon f h^2+\rho g)).$$

Using the prescription

$$B_{k+K-1}^{n+N-1}(gf) = \frac{1}{kN-Kn}\left(B_k^n(g)B_K^N(f) - B_K^N(f)B_k^n(g)\right)$$

and suitable linear combinations, we obtain the representation of the B_y^x (and therefore of the RHPWN and centreless Virasoro (or Witt)–Zamolodchikov–w_∞ commutation relations) on $\mathcal{F}$.

9. Classical stochastic processes on $\mathcal{F}_n$

Definition 6. *A quantum stochastic process $x = \{x(t)/t \geqslant 0\}$ is a family of Hilbert space operators. Such a process is said to be classical if for all $t, s \geqslant 0, x(t) = x(t)^*$ and $[x(t), x(s)] = x(t)x(s) - x(s)x(t) = 0$.*

Proposition 3. *Let a quantum stochastic process $x = \{x(t)/t \geqslant 0\}$ be defined by $x(t) = \sum_{n,k\in\Lambda} c_{n,k}B_k^n(t)$ where $c_{n,k} \in \mathbb{C} - \{0\}$, Λ is a finite subset of $\{0, 1, 2, \ldots\}$ and $B_k^n(t) = B_k^n(\chi_{[0,t]})$. If for each $n, k \in \Lambda$, $c_{n,k} = \bar{c}_{k,n}$ then the process $x = \{x(t)/t \geqslant 0\}$ is classical.*

Proof. By (2.2) $x(t) = x^*(t)$ for all $t \geqslant 0$. Moreover, by (2.4), $[x(t), x(s)] = 0$ for all $t, s \geqslant 0$ since each term of the form $c_{N,K}c_{n,k}\left[B_K^N(t), B_k^n(s)\right]$ is cancelled out by the corresponding term of the form $c_{n,k}c_{N,K}\left[B_k^n(s), B_K^N(t)\right]$. Thus the process $x = \{x(t)/t \geqslant 0\}$ is classical. □

In the remaining of this section we will study the classical process $x = \{x(t)/t \geqslant 0\}$ whose Fock representation as a family of operators on $\mathcal{F}_n$ is $x(t) = B_0^n(t) + B_n^0(t)$.

Lemma 4 (splitting formula)**.** *Let $s \in \mathbb{R}$ and let μ be as in notation 1. Then for $n = 1$*

$$\mathrm{e}^{s(B_0^1+B_1^0)}\Phi = \mathrm{e}^{\frac{s^2}{2}\mu}\,\mathrm{e}^{sB_0^1}\Phi$$

and for $n \geqslant 2$

$$\mathrm{e}^{s(B_0^n+B_n^0)}\Phi = \left(\sec\left(\sqrt{\frac{n^3(n-1)}{2}}s\right)\right)^{\frac{2n\mu}{n^3(n-1)}}\mathrm{e}^{\sqrt{\frac{2}{n^3(n-1)}}\tan\left(\sqrt{\frac{n^3(n-1)}{2}}s\right)B_0^n}\Phi.$$

Proof. We will use the 'differential method' of proposition 4.1.1, chapter 1 of [16]. So let

$$E\Phi = \mathrm{e}^{s(B_0^n+B_n^0)}\Phi = \mathrm{e}^{V(s)B_0^n}\,\mathrm{e}^{W(s)}\Phi, \tag{9.1}$$

where W, V are real-valued functions with $W(0) = V(0) = 0$. Then,

$$\frac{\partial}{\partial s} E\Phi = \left(B_0^n + B_n^0\right) E\Phi = B_0^n E\Phi + B_n^0 E\Phi. \tag{9.2}$$

By lemma 3 we have

$$\begin{aligned}
B_n^0 E\Phi &= B_n^0 \, \mathrm{e}^{V(s)B_0^n} \, \mathrm{e}^{W(s)} \Phi = \mathrm{e}^{W(s)} B_n^0 \, \mathrm{e}^{V(s)B_0^n} \Phi \\
&= \mathrm{e}^{W(s)} \sum_{k=0}^{\infty} \frac{V(s)^k}{k!} B_n^0 \left(B_0^n\right)^k \Phi \\
&= \mathrm{e}^{W(s)} \sum_{k=0}^{\infty} \frac{V(s)^k}{k!} nk \left(\mu + (k-1)\frac{n^2(n-1)}{2}\right) \left(B_0^n\right)^{k-1} \Phi \\
&= \left(n\mu V(s) + \frac{n^3(n-1)}{2} V(s)^2 B_0^n\right) \mathrm{e}^{V(s)B_0^n} \, \mathrm{e}^{W(s)} \Phi \\
&= \left(n\mu V(s) + \frac{n^3(n-1)}{2} V(s)^2 B_0^n\right) E\Phi.
\end{aligned}$$

Thus (9.2) becomes

$$\frac{\partial}{\partial s} E\Phi = \left(B_0^n + n\mu V(s) + \frac{n^3(n-1)}{2} V(s)^2 B_0^n\right) E\Phi. \tag{9.3}$$

From (9.1) we also have that

$$\frac{\partial}{\partial s} E\Phi = \left(V'(s) B_0^n + W'(s)\right) E\Phi. \tag{9.4}$$

From (9.3) and (9.4), by equating coefficients of 1 and B_0^n, we have that

$$W'(s) = n\mu V(s) \tag{9.5}$$

$$V'(s) = 1 + \frac{n^3(n-1)}{2} V(s)^2 \text{ (Riccati equation).} \tag{9.6}$$

For $n = 1$ we find that $V(s) = s$ and $W(s) = \frac{s^2}{2}\mu$. For $n \geqslant 2$ by separating the variables we find that

$$V(s) = \sqrt{\frac{2}{n^3(n-1)}} \tan\left(\sqrt{\frac{n^3(n-1)}{2}} s\right)$$

and so

$$W(s) = -\frac{2n\mu}{n^3(n-1)} \ln\left(\cos\left(\sqrt{\frac{n^3(n-1)}{2}} s\right)\right)$$

which implies that

$$\mathrm{e}^{W(s)} = \left(\sec\left(\sqrt{\frac{n^3(n-1)}{2}} s\right)\right)^{\frac{2n\mu}{n^3(n-1)}}$$

thus completing the proof. □

In the theory of Bernoulli systems and the Fock representation of finite-dimensional Lie algebras (cf chapter 5 of [16]) the Riccati equation (9.6) has the general form

$$V'(s) = 1 + 2\alpha V(s) + \beta V(s)^2$$

and the values of α and β determine the underlying classical probability distribution and the associated special functions. For example, for $\alpha = 1 - 2p$ and $\beta = -4pq$ we have the binomial process and the Krawtchouk polynomials, for $\alpha = p^{-1} - \frac{1}{2}$ and $\beta = qp^{-2}$ we have the negative binomial process and the Meixner polynomials, for $\alpha \neq 0$ and $\beta = 0$ we have the Poisson process and the Poisson–Charlier polynomials, for $\alpha^2 = \beta$ we have the exponential process and the Laguerre polynomials, for $\alpha = \beta = 0$ we have Brownian motion with moment generating function $\mathrm{e}^{\frac{s^2}{2}t}$ and associated special functions the Hermite polynomials, and for $\alpha^2 - \beta < 0$ we have the continuous binomial and beta processes (cf chapter 5 of [16] and also [17]) with moment generating function $(\sec s)^t$ and associated special functions the Meixner–Pollaczek polynomials. In the infinite-dimensional TRHPWN case the underlying classical probability distributions are given in the following.

Proposition 4 (moment generating functions)**.** *For all $s \geqslant 0$*

$$\left\langle \mathrm{e}^{s(B_0^1(t)+B_1^0(t))}\right\rangle_1 := \left\langle \mathrm{e}^{s(B_0^1(t)+B_1^0(t))}\Phi, \Phi\right\rangle_1 = \mathrm{e}^{\frac{s^2}{2}t}$$

i.e. $\left\{B_0^1(t) + B_1^0(t)/t \geqslant 0\right\}$ *is Brownian motion (cf [16], [18]) while for $n \geqslant 2$*

$$\langle \mathrm{e}^{s(B_0^n(t)+B_n^0(t))}\rangle_n := \langle \mathrm{e}^{s(B_0^n(t)+B_n^0(t))}\Phi, \Phi\rangle_n = \left(\sec\left(\sqrt{\frac{n^3(n-1)}{2}}s\right)\right)^{\frac{2nt}{n^3(n-1)}}$$

i.e. $\{B_0^n(t) + B_n^0(t)/t \geqslant 0\}$ *is for each n a continuous binomial/beta process (see section 10 below).*

Proof. The proof follows from lemma 4, $\mu([0, t]) = t$, and the fact that for all $n \geqslant 1$ we have $B_n^0(t)\Phi = 0$. □

10. The continuous binomial and beta processes

Let $b_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}$, where $n, k \in \{0, 1, 2, \ldots\}$, $n \geqslant k$ and $x \in (0, 1)$, be the standard binomial distribution. Using the gamma function we can analytically extend from $n, k \in \{0, 1, 2, \ldots\}$ to $z, w \in \mathbb{C}$ with $\mathrm{Re}\, z \geqslant \mathrm{Re}\, w > -1$ and we have

$$\begin{aligned} b_{z,w}(x) &= \frac{\Gamma(z+1)}{\Gamma(z-w+1)\Gamma(w+1)}x^w(1-x)^{z-w} \\ &= \frac{1}{z+1}\frac{1}{B(z-w+1, w+1)}x^{(w+1)-1}(1-x)^{(z-w+1)-1} = \frac{1}{z+1}\beta_{w+1,z-w+1}(x) \end{aligned}$$

where for $\mathrm{Re}\, a > 0$ and $\mathrm{Re}\, c > 0$, $B(a, c) = \frac{\Gamma(a)\Gamma(c)}{\Gamma(a+c)} = \int_0^1 x^{a-1}(1-x)^{c-1}\,\mathrm{d}x$ is the beta function and $\beta_{w+1,z-w+1}$ is the analytic continuation to $\mathrm{Re}\, a > 0$ and $\mathrm{Re}\, c > 0$ of the standard beta distribution $\beta_{a,c}(x) = \frac{\Gamma(a+c)}{\Gamma(a)\Gamma(c)}x^{a-1}(1-x)^{c-1}$ where $a > 0$ and $c > 0$.

Proposition 5. *For each $t > 0$ let X_t be a random variable with distribution given by the density*

$$p_t(x) = \frac{2^{t-1}}{2\pi}B\left(\frac{t+ix}{2}, \frac{t-ix}{2}\right),$$

where B is the beta function. Then the moment generating function of X_t is

$$\langle \mathrm{e}^{sX_t}\rangle = \int_{-\infty}^{\infty} \mathrm{e}^{sx}p_t(x)\,\mathrm{d}x = (\sec s)^t;\ s \in \mathbb{R}. \qquad (10.1)$$

Proof. See proposition 4.1.1, chapter 5 of [16]. □

Corollary 5. *With X_t and p_t as in proposition 5, let $Y_t = \sqrt{\frac{n^3(n-1)}{2}} X_t$. Then the moment generating function of Y_t with respect to the density $q_t = p_{\frac{2n}{n^3(n-1)}t}$, where $n \in \{2, 3, \ldots\}$, is*

$$\langle \mathrm{e}^{sY_t} \rangle = \left(\sec \left(\sqrt{\frac{n^3(n-1)}{2}} s \right) \right)^{\frac{2nt}{n^3(n-1)}}.$$

Proof. For each $t > 0$, p_t is a probability density function. Therefore $\int_{-\infty}^{\infty} p_t(x)\,\mathrm{d}x = 1$. Replacing t by $\frac{2n}{n^3(n-1)}t$ we obtain $\int_{-\infty}^{\infty} p_{\frac{2n}{n^3(n-1)}t}(x)\,\mathrm{d}x = 1$ and so $\int_{-\infty}^{\infty} q_t(x)\,\mathrm{d}x = 1$ which means that for each $t > 0$, q_t is a probability density function. Replacing t by $\frac{2n}{n^3(n-1)}t$ and s by $\sqrt{\frac{n^3(n-1)}{2}} s$ in (10.1) we obtain

$$\int_{-\infty}^{\infty} \mathrm{e}^{s\sqrt{\frac{n^3(n-1)}{2}}x} q_t(x)\,\mathrm{d}x = \left(\sec \left(\sqrt{\frac{n^3(n-1)}{2}} s \right) \right)^{\frac{2nt}{n^3(n-1)}}$$

which is precisely the moment generating function $\langle \mathrm{e}^{sY_t} \rangle$ of Y_t with respect to q_t. □

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